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On Browder Spectrum and Quasisimilarity of M-hyponormal Operators in Hilbert Spaces

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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ABSTRACT

The spectrum of operators and quasisimilarity of m-hyponormal operators in Hilbert Spaces has been extensively studied. It has been shown that quasisimilarity of m-hyponormal operators have equal spectrum, equal Essential Spectrum and equal Weyl Spectrum. However, the consideration of equality of the Browder spectrum for m-hyponormal operators together with quasisimilarity has not been fully determined. Therefore, this study aims to consider the quasisimilarity of m-hyponormal operators and determine the conditions under which quasisimilarity of m-hyponormal operators yields equal Browder spectrum.

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1. INTRODUCTION

The concept of quasisimilarity was first introduced by Nagy and Foias [1] in the theory of infinite-dimensional analogue of the Jordan form for certain classes of contractions to study invariant subspace structures. Nagy and Foias [1] thus defined the quasisimilar operators as: Two operators $A \in B(H)$ and $B \in B(K)$ are said to be quasisimilar if there exist quasi-affinity operators X from K and Y from H to K which satisfy the equation; $XA = BX$ and $AY = YB$. In all classes of operators, quasisimilarity is an equivalence relation. Clary [2] showed that quasisimilarity is the same thing as similarity in finite-dimensional spaces, but in infinite-dimensional spaces, it is a much weaker relation, so weak that two operators can be quasisimilar and yet they have no equal spectra.

Quasisimilar operators on finite-dimensional spaces have equal spectra but in the case of infinite-dimensional Hilbert spaces, Nagy and Foias [1] proved that the operators may be quasisimilar but have spectra which are not equal. Clary [2] showed the conditions under which two quasisimilar transformations will be having equal spectra, that is, if the operators are hyponormal. Clary [2] also showed that two operators which are quasisimilar and hyponormal operators have the same spectra and posed the question whether they have equal essential spectra. Williams [3] extended the work of Clary [2] to determine whether quasisimilar hyponormal operators have equal essential spectra and showed that that quasisimilar quasinormal operators have equal spectra. Williams [3] also gave some conditions under which quasisimilar hyponormal operators have equal essential spectra, that is, if the two operators are both hyponormal, partial isometry or are quasinormal, then they have equal essential spectra. Yang [4] showed that the operators which are quasisimilar and m-hyponormal operators have the same essential spectrum. Khalagai and Nyamai [5] extended the work of Yang [4] and showed that proved that the operators which are quasisimilar m-hyponormal operators have equal spectra. Maina [6] showed that quasisimilar m-hyponormal together with biquasitriangular operator have equal Weyl spectrum, that is, if they are quasisimiliar m-hyponormal and biquasitriangular operators. However, it has not been shown whether the quasisimilarity of m-hyponormal operators have equal Browder spectrum. Therefore, the purpose of this study was to consider the quasisimilarity of m-hyponormal operators and determine the conditions under which quasisimilarity of m-hyponormal operators yields equal Browder spectrum**.**

2. DEFINITIONS AND TERMINOLOGIES

Definition 2.1. Hilbert space [7]: A Hilbert space H is a complete inner product space.

Definition 2.2. Normal operator [8]: An operator $T \in (H)$ is said to be normal if $TT^{*}=T^{*}T$

Definition 2.3. Hyponormal operator [9]: An operator $T \in (H)$ is said to be hyponormal if $T^{*}T \geq TT^{*}$

Definition 2.4. M-hyponormal operator [10]: An operator $T \in (H)$ is said to be M-hyponormal if there exist a real number M such that

$$\|(T-\lambda)^{*}f\| \leq M\|(T-\lambda)f\|,\ \forall f \in H, \lambda \in \mathbb{C}.$$

Definition 2.5. Spectrum [11]: Let $T \in (H)$, then the set $\delta(T) = \{\lambda \in \mathbb{C}: (T-\lambda I)$ is not invertible$\}$ is called the spectrum of T whereas the complement of the spectrum of T is called the resolvent of T.

Definition 2.6. Essential spectrum [12]: Let $T \in (H)$, the essential spectrum denoted by $\delta_e T$ is defined by,

$\delta_e T = \{\lambda \in \mathbb{C}: (T-\lambda I)$ is not Fredholm operator$\}$

Definition 2.7. Weyl spectrum [13]: Let $T \in (H)$, the Weyl spectrum denoted by $\delta_w T$ is defined by,

$\delta_w T = \{\lambda \in \mathbb{C}: (T-\lambda I)$ is not Weyl operator$\}$

Definition 2.8. Browder spectrum [14]: Let $T \in (H)$, the Browder spectrum denoted by $\delta_b T$ is defined by,

$\delta_b T = \{\lambda \in \mathbb{C}: (T-\lambda I)$ is not Browder operator$\}$

Remark 2.9. Maina, [6] showed that Essential spectrum, Weyl spectrum, Browder spectrum and Spectrum of operators form a nested type of set, that is,

Essential spectrum ⊂ Weyl spectrum ⊂ Browder spectrum

3. METHODOLOGY

The properties of normal and hyponormal operators, biquasitriangular and spectral properties of classes of operators together with Browder's Theorem were considered.

The following properties were useful

Preposition 3.1. [15]: A bounded linear operator T is called Browder if it is Fredholm of finite ascent and descent. The essential spectrum δ_e (T), Weyl spectrum δ_w (T) and Browder spectrum δ_b (T) of $T \in L(X)$ are defined by

$\delta e(T) = \{ \lambda \in \mathbb{C} : (T - I\lambda)$ is not Fredholm operator}

$\delta w(T) = \{ \lambda \in \mathbb{C} : (T - I\lambda)$ is not Weyl operator}

$\delta b(T) = \{ \lambda \in \mathbb{C} : (T - I\lambda)$ is not Browder operator}

Evidently, $\delta e(T) \subseteq \delta w(T) \subseteq \delta_b(T)$

Lemma 3.2. [15]: The Weyl's theorem holds for $T \in L(X)$ if $\delta(T) \setminus \delta w(T) = \pi_{00}(T)$ where $\pi_{00}(T)$ is the set of isolated points of $\delta(T)$ which are eigenvalues of finite multiplicity, that is,

$\pi_{00}(T) = \{ \lambda \in iso\ \delta(T) : 0 < \dim N(T - I\lambda) < \infty\}$

Lemma 3.3. [15]: The Browder's theorem holds for $T \in L(X)$ if $\delta_w(T) = \delta_b(T)$

Alternatively, the Browder's theorem holds for $T \in L(X)$ if $\delta(T) = \delta_w(T) \cup \pi_{00}(T)$

Lemma 3.4. [16]: Weyl's theorem holds for m-hyponormal

Preposition 3.5. [15]: For any $T \in L(X)$, then the following implications holds

i) a-Weyl's theorem ⟹ Weyl's theorem ⇒ Browder's theorem
ii) a-Weyl's theorem⟹a-Browder's theorem⇒ Browder's theorem

Lemma 3.6. [6]: Let $A, B \in B(H)$ be quasisimilar M-hyponormal and biquasitriangular operators. Then $\delta_w A = \delta_w B$.

4. RESULTS

The following results were established

Theorem 4.1: Let $A, B \in B(H)$ be quasisimilar operators. If an operator A is a normal operator, then an operator B is also a normal operator.

Proof:

since A and B are quasisimilar then there exist two operators F and G that are quasi-affinity/ quaisi-invertible such that

$$AF = FB \text{ and } BG = GB$$

Considering $AF = FB$ … … … … .(i)

Since F is quasi-invertible then by multiplying equation (i) from the right by F^{-1} we have

$$AFF^{-1} = FBF^{-1}$$

Implies that

$$A = FBF^{-1} \text{ … … … … … … … … … … … } \text{….}(ii)$$

Which follows that

$$A^* = FB^*F^{-1} \text{ … … … … … … … } (iii)$$

Since operator A is normal, then by the definition

$$A^*A = AA^* \text{ … … … … … … … … … … … } \text{…..}(iv)$$

Substituting the equations (ii) and (iii) into equation (iv)

$$FB^*F^{-1}FBF^{-1} = BF^{-1}FB^*F^{-1}$$

But $F^{-1}F = I$, thus

$$FB^*IBF^{-1} = BIB^*F^{-1}$$

$$FB^*BF^{-1} = FBB^*F^{-1} \text{ … … … … … … … }(v)$$

Multiplying equation (v) from left by F^{-1} and multiplying from the right side by F, we have

$$F^{-1}FB^*BF^{-1}F = F^{-1}FBB^*F^{-1}F$$

$$B^*B = BB^*$$

By the definition of normal operator, it implies that B is normal operator ■

Theorem 4.2: Let $P, Q \in B(H)$ be quasisimilar operators. If an operator P is a hyponormal operator then an operator Q is also a hyponormal.

Proof:

since P and Q are quasisimilar then there exist two operators X and Y that are quasi-affinity/ quaisi-invertible such that

$$PX = XQ \text{ and } PY = YQ$$

Considering $PX = XQ \dots\dots\dots \qquad \dots.(i)$

Since X is quasi-invertible then multiplying equation (i) from right by X^{-1} we have

$$PXX^{-1} = XPX^{-1}$$

Implies that $P = XQX^{-1} \dots \qquad .(ii)$

Which follows that $P^* = XQ^*X^{-1} \dots \qquad \dots(iii)$

Since operator P is hyponormal, then by the definition

$$P^*P \geq PP^* \dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \dots\dots \dots..(iv)$$

Substituting the equations (ii) and (iii) into equation (iv)

$$XQ^*X^{-1}XQX^{-1} \geq XQX^{-1}XQ^*X^{-1}$$

But $X^{-1}X = I$, thus

$$XQ^*IQX^{-1} \geq XQIQ^*X^{-1}$$

$$XQ^*QX^{-1} \geq XQQ^*X^{-1} \dots\dots\dots\dots\dots\dots\dots(v)$$

multiplying equation (v) from left by X^{-1} and multiplying from the right side by X, we have

$$X^{-1}XQ^*QX^{-1}X \geq X^{-1}XQQ^*X^{-1}X$$

$$Q^*Q \geq QQ^*$$

Which implies that Q is hyponormal operator. ■

Theorem 4.3: Let $U, V \in B(H)$ be quasisimilar operators. If an operator U is a m-hyponormal operator then an operator V is also a m-hyponormal.

Proof:

since U and V are quasisimilar then there exist two operators S and T that are quasi-affinity/ quaisi-invertible such that

$$US = SV \text{ and } UT = TV$$

Considering $US = SV \dots\dots\dots\dots. \qquad (i)$

Since S is quasi-invertible then multiplying equation (i) from right by S^{-1} we have

$$USS^{-1} = SVS^{-1}$$

Implies that $U = SVS^{-1}$

It follows that for any $\lambda \in \mathbb{R}$, then

$$(U - \lambda I) = S(V - \lambda I)S^{-1} \qquad \dots\dots\dots..(ii)$$

Which implies that

$$(U - \lambda I)^* = S(V - \lambda I)^*S^{-1}. \qquad (iii)$$

And since the operator U is m-hyponormal, then by the definition

$\| (U - \lambda I)^* \| \leq M \| (U - \lambda I)y \| \ \forall y \in H$ and $\lambda \in \mathbb{C}$

But $\| (U - \lambda I)^* \| = < (U - \lambda I)^*y, (U - \lambda I)^*y >^{\frac{1}{2}}$
$= < (U - \lambda I)(U - \lambda I)^*y, y >^{\frac{1}{2}} \qquad \dots\dots\dots\dots\dots.(a)$

And $\| (U - \lambda I)y \| = < (U - \lambda I)y, (U - \lambda I)y >^{\frac{1}{2}}$
$= < (U - \lambda I)^*(U - \lambda I)y, y >^{\frac{1}{2}} \dots\dots\dots\dots\dots(b)$

Thus $< (U - \lambda I)(U - \lambda I)^*y, y >^{\frac{1}{2}} = \| (U - \lambda I)^* \| \leq M \| (U - \lambda I)y \|$

$$\leq M < (U - \lambda I)^*(U - \lambda I)y, y >^{\frac{1}{2}}$$

Therefore

$< (U - \lambda I)(U - \lambda I)^*y, y >^{\frac{1}{2}} \leq M < (U - \lambda I)^*(U - \lambda I)y, y >^{\frac{1}{2}} \dots\dots\dots\dots \qquad \dots(iv)$

Substituting the equations (ii) and (iii) into equation (iv)

$$< S(V - \lambda I)S^{-1}S(V - \lambda I)^*S^{-1}y, y >^{\frac{1}{2}} = < (U - \lambda I)(U - \lambda I)^*y, y >^{\frac{1}{2}}$$

$$\leq M \| (U - \lambda I)y \|$$

$$\leq M < (U - \lambda I)y, (U - \lambda I)y >^{\frac{1}{2}}$$

$$\leq M < (U - \lambda I)^*(U - \lambda I)y, y >^{\frac{1}{2}}$$

$$\leq M < S(V - \lambda I)^*S^{-1} S(V - \lambda I)S^{-1}y, y >^{\frac{1}{2}}$$

Thus

$$< S(V-\lambda I)S^{-1}S(V-\lambda I)^{*}S^{-1}y, y >^{\frac{1}{2}} \leq M < S(V-\lambda I)^{*}S^{-1}S(V-\lambda I)S^{-1}y, y >^{\frac{1}{2}}$$

But $S^{-1}S = I$, implies that

$$< S(V-\lambda I)(V-\lambda I)^{*}S^{-1}y, y >^{\frac{1}{2}} \leq M < S(V-\lambda I)^{*}(V-\lambda I)S^{-1}y, y >^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots (v)$$

multiplying equation (v) from left by S^{-1} and multiplying from the right side by S, we have

$$< S^{-1}S(V-\lambda I)(V-\lambda I)^{*}S^{-1}Sy, y >^{\frac{1}{2}} \leq M < S^{-1}S(V-\lambda I)^{*}(V-\lambda I)S^{-1}Sy, y >^{\frac{1}{2}}$$

$$< (V-\lambda I)(V-\lambda I)^{*}y, y >^{\frac{1}{2}} \leq M < (V-\lambda I)^{*}(V-\lambda I)y, y >^{\frac{1}{2}}$$

From equations (a) and (b) we have

$$\| (V-\lambda I)^{*} \| \leq M \| (V-\lambda I)y \|$$

Which implies that an operator V is m-hyponormal operator ■

Theorem 4.4: Let $P, Q \in B(H)$ be quasisimilar m-hyponormal and biquasitriangular operators. Then δ(P)= δ_b(Q).

Proof:

Since P and Q are quasisimilar m-hyponormal operators, it implies that

δ_e(P)= δ_e(Q)

but P and Q are biquasitriangular, then it implies that

δ_e(P)= δ_w(P) and δ_e(Q)= δ_w(Q), thus

δ_w(P)= δ_e(P)= δ_e(Q) =δ_w(Q) … … … … … (i)

But since P and Q are quasisimilar and biquasitriangular, then it implies that they have Weyl spectrum which are equal. By lemma 3.4, Weyl's theorem holds for m-hyponormal operator and since P and Q are both m-hyponormal operators, it implies that Weyl's theorem holds for both P and Q such that

$\delta(\mathrm{P}) \setminus \delta\mathrm{w}(\mathrm{P}) = \pi_{00}$(P)

$\delta(\mathrm{Q}) \setminus \delta\mathrm{w}(\mathrm{Q}) = \pi_{00}$(Q)

by the preposition 3.1, it implies that

Weyl's theorem ⇒ Browder's theorem, which means that all operators that satisfy the Weyl's theorem will also satisfy the Browder's theorem however the converse is not necessarily true.

Thus, Weyl's theorem holds for both P and Q, then it implies that Browder's theorem also holds for the operators P and Q. By the lemma 3.3, it implies that

δ_w(P)= δ_b(P) and δ_w(Q)= δ_b(Q)

From equation (i), we have

δ_w(P)= δ_e(P)= δ_e(Q) =δ_w(Q) but δ_w(P)= δ_b(P) and δ_w(Q)= δ_b(Q)

Thus,

δ_w(P)= δ_b(P)= δ_e(P)= δ_e(Q) =δ_w(Q)= δ_b(Q)

Which implies that

δ_b(P) = δ_b(Q) ■

5. CONCLUSION

From the preceding results, it establishes that quasisimilarity preserves the normality, hyponormality and m-hyponormality. It also establishes that that quasisimilar m-hyponormal operators have equal Browder spectrum.

DISCLAIMER (ARTIFICIAL INTELLIGENCE)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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